Unilateral global bifurcation and nodal solutions for the p-Laplacian with sign-changing weight *

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Abstract

In this paper, we shall establish a Dancer-type unilateral global bifurcation result for a class of quasilinear elliptic problems with sign-changing weight. Under some natural hypotheses on perturbation function, we show that $(\mu_k^\nu(p), 0)$ is a bifurcation point of the above problems and there are two distinct unbounded continua, $(\mathcal{C}_k^\nu)^+$ and $(\mathcal{C}_k^\nu)^-$, consisting of the bifurcation branch $\mathcal{C}_k^\nu$ from $(\mu_k^\nu(p), 0)$, where $\mu_k^\nu(p)$ is the k-th positive or negative eigenvalue of the linear problem corresponding to the above problems, $\nu \in \{+, -\}$. As the applications of the above unilateral global bifurcation result, we study the existence of nodal solutions for a class of quasilinear elliptic problems with sign-changing weight. Moreover, based on the bifurcation result of Drábek and Huang (1997) [11], we study the existence of one-sign solutions for a class of high dimensional quasilinear elliptic problems with sign-changing weight.

Keywords: p-Laplacian; Unilateral global bifurcation; Nodal solutions; sign-changing weight

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1 Introduction

In [28], Rabinowitz established Rabinowitz's unilateral global bifurcation theory. However, as pointed out by Dancer [6, 7] and López-Gómez [22], the proofs of these theorems contain gaps. Fortunately, Dancer [6] gave a corrected version of unilateral global bifurcation theorem. In 1997, Drábek and Huang [11] proved a Dancer-type bifurcation theorem (Theorem 4.5, [11]) in which the continua bifurcated from the principle eigenvalue for a high dimension p-Laplacian problem with sign-changing weight in $\mathbb{R}^N$. However, no any information on the high eigenvalue for p-Laplacian problem with sign-changing weight. For the case of definite weight, Dai and Ma [5] established a Dancer-type unilateral global bifurcation result for one-dimensional p-Laplacian. In [16], Girg and Takáč proved a Dancer-type bifurcation theorem for a high dimensional p-Laplacian equation.

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It is the main purpose of this paper to establish a similar result to Dai and Ma's about the continua of radial solutions for the following N-dimensional p-Laplacian problem on the unit ball of $\mathbb{R}^N$ with $N \geq 1$ and $1 < p < +\infty$,

$$\begin{cases} -\text{div}\,(\varphi_p(\nabla u)) = \mu m(x)\varphi_p(u) + g(x, u; \mu), & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases} \tag{1.1}$$

where B is the unit open ball of $\mathbb{R}^N$, $\varphi_p(s) = |s|^{p-2}s$, $m \in M(B)$ is a sign-changing function with

$$M(B) = \left\{ m \in C(\overline{B}) \text{ is radially symmetric } \big| \text{meas}\{x \in B, m(x) > 0\} \neq 0 \right\},$$

$g : B \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition in the first two variables and radially symmetric with respect to x.

It is clear that the radial solutions of (1.1) is equivalent to the solutions of the following problem

$$\begin{cases} -\left(r^{N-1}\varphi_p(u')\right)' = \mu m(r)r^{N-1}\varphi_p(u) + r^{N-1}g(r, u; \mu), & \text{a.e. } r \in (0, 1), \\ u'(0) = u(1) = 0, \end{cases} \tag{1.2}$$

where $r = |x|$ with $x \in B$, $m \in M(I)$ is a sign-changing with $I = (0, 1)$ and

$$M(I) = \left\{ m \in C(\overline{I}) \text{ is radially symmetric } \big| \text{meas}\{r \in I, m(r) > 0\} \neq 0 \right\}.$$

We also assume the perturbation function g satisfies the following hypotheses:

$$\lim_{s \to 0} \frac{g(r, s; \mu)}{|s|^{p-1}} = 0 \tag{1.3}$$

uniformly for a.e. $r \in I$ and μ on bounded sets.

Under the condition of $m \in M(I)$ and (1.3), we shall show that $(\mu_k^\nu(p), 0)$ is a bifurcation point of (1.2) and there are two distinct unbounded continua, $(\mathcal{C}_k^\nu)^+$ and $(\mathcal{C}_k^\nu)^-$, consisting of the bifurcation branch $\mathcal{C}_k^\nu$ from $(\mu_k^\nu(p), 0)$, where $\mu_k^\nu(p)$ is the k-th positive or negative eigenvalue of the linear problem corresponding to (1.2), where $\nu \in \{+, -\}$.

Based on the unilateral global bifurcation result (see Theorem 3.2), we investigate the existence of radial nodal solutions for the following p-Laplacian problem

$$\begin{cases} -\text{div}\,(\varphi_p(\nabla u)) = \gamma m(x)f(u), & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases} \tag{1.4}$$

where $f \in C(\mathbb{R})$, γ is a parameter.

It is clear that the radial solutions of (1.4) is equivalent to the solutions of the following problem

$$\begin{cases} \left(r^{N-1}|u'|^{p-2}u'\right)' + \gamma r^{N-1}m(r)f(u) = 0, & r \in I, \\ u'(0) = u(1) = 0, \end{cases} \tag{1.5}$$

where $r = |x|$ with $x \in B$.

It is well known that when $m(r) \equiv 1$ and $f(r, u) = \lambda \varphi_p(u)/\gamma$, problem (1.5) has a nontrivial solution if and only if λ is an eigenvalue of the following problem

$$\begin{cases} \left(r^{N-1}|u'|^{p-2}u'\right)' + \lambda r^{N-1}\varphi_p(u) = 0, & r \in I, \\ u'(0) = u(1) = 0. \end{cases} \tag{1.6}$$

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In particular, when $\lambda = \lambda_k(p)$, there exist two solutions u_k^+ and u_k^-, such that u_k^+ has exactly $k-1$ zeros in I and is positive near 0, and u_k^- has exactly $k-1$ zeros in I and is negative near 0 (see [27], Theorem 1.5.3).

When $p = 2$, $N = 1$ and $m(r) \geq 0$, Ma and Thompson [24] considered the interval of γ, in which there exist nodal solutions of (1.5) under some suitable assumptions on f. The results of the above have been extended to the the case of weight function changes its sign by Ma and Han [23]. The results they obtained extended some well known theorems of the existence of positive solutions for related problems [12, 13, 17] and sign-changing solutions [26]. For the case $p \not\equiv 2$ but $N = 1$, $m(r) \geq 0$, Dai and Ma [5] proved the existence of nodal solutions for (1.5).

However, few results on the existence of radial nodal solutions, even positive solutions, have been established for N-dimensional p-Laplacian problem with sign-changing weight $m(r)$ on the unit ball of $\mathbb{R}^N$. In this paper, we shall establish a similar result to Ma and Thompson [24] for N-dimensional p-Laplacian problem with sign-changing weight. Problem with sign-changing weight arises from the selection-migration model in population genetics. In this model, $m(r)$ changes sign corresponding to the fact that an allele A_1 holds an advantage over a rival allele A_2 at same points and is at a disadvantage at others; the parameter r corresponds to the reciprocal of diffusion, for detail, see [15]. For the applications of nodal solutions, see Lazer and McKenna [19] and Kurth [18].

In high dimensional general domain case, based on Drábek and Huang's results (note their results also valid for bounded smooth domain), we shall investigate the existence of one-sign solutions for the problem (1.4) with $1 < p < N$ and the general smooth domain $\Omega \subset \mathbb{R}^N$ with $N \geq 2$, i.e.,

$$\begin{cases} -\operatorname{div}(\varphi_p(\nabla u)) = \gamma m(x) f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

By a solution of (1.7) we understand $u \in W_0^{1,p}(\Omega)$ satisfying (1.7) in the weak sense.

The rest of this paper is arranged as follows. In Section 2, we establish the eigenvalue theory of second order p-Laplacian Dirichlet boundary value problem in the radial case with sign-changing weight. In Section 3, we establish the unilateral global bifurcation theory for (1.2). In Section 4, we prove the existence of nodal solutions for (1.5). In Section 5, we study the existence of one-sign solutions for (1.7).

2 Some preliminaries

In [25], by Prüfer transformation, Meng, Yan and Zhang established the spectrum of one-dimensional p-Laplacian with an indefinite integrable weight. When $m \equiv 1$, using oscillation method, Peral [27] established the eigenvalue theory of N-dimensional p-Laplacian on the unite ball. However, applying their methods to N-dimensional p-Laplacian on the unit ball with indefinite weight is very difficult, even cann't be used. In [2], using variational method, Anane, Chakrone and Monssa established the spectrum of one-dimensional p-Laplacian with an indefinite weight. While, we don't know whether or not the eigenvalue function $\mu_k^\nu(p)$ is continuous with respect to p, which was obtained by Anane, Chakrone and Monssa. In this Section, by similar method of Anane, Chakrone and Monssa's, we can establish the eigenvalue theory of second order p-Laplacian Dirichlet boundary value problem in the radial case with indefinite weight. It is well known the continuity of eigenvalues with respect to p is very important in the studying of the global

bifurcation phenomena for p-Laplacian problems, see [9, 10, 20, 27]. In this Section, we shall also show that $\mu_k^\nu(p)$ is continuous with respect to p. Moreover, we also establish a key lemma which will be used in Section 4.

Applying the similar method to prove [2, Theorem 1] with obvious changes, we can obtain the following:

Theorem 2.1. *Assume $m \in M(I)$. The eigenvalue problem*

$$\begin{cases} \left(r^{N-1}|u'|^{p-2}u'\right)' + \mu m(r)r^{N-1}|u|^{p-2}u = 0, \ r \in I, \\ u'(0) = u(1) = 0 \end{cases} \tag{2.1}$$

has two infinitely many simple real eigenvalues

$$0 < \mu_1^+(p) < \mu_2^+(p) < \cdots < \mu_k^+(p) < \cdots, \ \lim_{k \to +\infty} \mu_k^+(p) = +\infty,$$

$$0 > \mu_1^-(p) > \mu_2^-(p) > \cdots > \mu_k^-(p) > \cdots, \ \lim_{k \to +\infty} \mu_k^-(p) = -\infty$$

and no other eigenvalues. Moreover,

1. Every eigenfunction corresponding to eigenvalue $\mu_k^\nu(p)$, has exactly $k-1$ zeros.

2. For every k, $\mu_k^\nu(m)$ verifies the strict monotonicity property with respect to the weight m.

Remark 2.1. Using Gronwall inequality [14], we can easily show that all zeros of eigenfunction corresponding to eigenvalue $\mu_k^\nu(p)$ is simple.

We first show that the principle eigenvalue function $\mu_1^\nu : (1, +\infty) \to \mathbb{R}$ is continuous.

Theorem 2.2. *The eigenvalue function $\mu_1^\nu : (1, +\infty) \to \mathbb{R}$ is continuous.*

Proof. The proof is similar to the proof of [9], but we give a rough sketch of the proof for reader's convenience. We only show that $\mu_1^+ : (1, +\infty) \to \mathbb{R}$ is continuous since the case of μ_1^- is similar. In the following proof, we shall shorten μ_1^+ to μ_1.

From the variational characterization of $\mu_1(p)$ it follows that

$$\mu_1(p) = \sup\left\{ \mu > 0 \Big| \mu \int_B m(x)|u|^p \, dx \le \int_B |\nabla u|^p \, dx, \text{ for all } u \in C_{r,c}^\infty(B) \right\}, \tag{2.2}$$

where $C_{r,c}^\infty(B) = \left\{ u \in C_c^\infty(B) \big| u \text{ is radially symmetric} \right\}$.

Let $\{p_j\}_{j=1}^\infty$ be a sequence in $(1, +\infty)$ convergent to $p > 1$. We shall show that

$$\lim_{j \to +\infty} \mu_1(p_j) = \mu_1(p). \tag{2.3}$$

To do this, let $u \in C_{r,c}^\infty(I)$. Then, from (2.2),

$$\mu_1(p_j) \int_B m(x)|u|^{p_j} \, dx \le \int_B |\nabla u|^{p_j} \, dx.$$

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On applying the Dominated Convergence Theorem we find

$$\limsup_{j \to +\infty} \mu_1(p_j) \int_B m(x)|u|^p \, dx \leq \int_B |\nabla u|^p \, dx. \tag{2.4}$$

Relation (2.4), the fact that u is arbitrary and (2.2) yield

$$\limsup_{j \to +\infty} \mu_1(p_j) \leq \mu_1(p).$$

Thus, to prove (2.3) it suffices to show that

$$\liminf_{j \to +\infty} \mu_1(p_j) \geq \mu_1(p). \tag{2.5}$$

Let $\{p_k\}_{k=1}^\infty$ be a subsequence of $\{p_j\}_{j=1}^\infty$ such that $\lim_{k \to +\infty} \mu_1(p_k) = \liminf_{j \to +\infty} \mu_1(p_j)$.

Let us fix $\varepsilon_0 > 0$ so that $p - \varepsilon_0 > 1$ and for each $0 < \varepsilon < \varepsilon_0$, $W_{r,0}^{1,p-\varepsilon}(B)$ is compactly embedded into $L_r^{p+\varepsilon}(B)$, here $W_{r,0}^{1,p-\varepsilon}(B) = \{u \in W_0^{1,p-\varepsilon}(B) | u$ is radially symmetric$\}$, $L_r^{p+\varepsilon}(B) = \{u \in L_r^{p+\varepsilon}(B) | u$ is radially symmetric$\}$. For $k \in \mathbb{N}$, let us choose $u_k \in W_{r,0}^{1,p_k}(B)$ such that

$$\int_B |\nabla u_k|^{p_k} \, dx = 1 \tag{2.6}$$

and

$$\int_B |\nabla u_k|^{p_k} \, dx = \mu_1(p_k) \int_B m(x)|u_k|^{p_k} \, dx. \tag{2.7}$$

For $0 < \varepsilon < \varepsilon_0$, there exists $k_0 \in \mathbb{N}$ such that $p - \varepsilon < p_k < p + \varepsilon$ for any $k \geq k_0$. Thus, for $k \geq k_0$, (2.6) and Hölder's inequality imply that

$$\int_B |\nabla u_k|^{p-\varepsilon} \, dx \leq |B|^{\frac{p_k - p + \varepsilon}{p_k}}, \tag{2.8}$$

where $|B|$ denotes the measure of B. This shows that $\{u_k\}_{k=k_0}^\infty$ is a bounded sequence in $W_{r,0}^{1,p-\varepsilon}(B)$. Passing to a subsequence if necessary, we can assume that $u_k \rightharpoonup u$ in $W_{r,0}^{1,p-\varepsilon}(B)$ and hence that $u_k \to u$ in $L_r^{p+\varepsilon}(B)$. Furthermore, $u \in L_r^p(B)$ and $u_k \to u$ in $L_r^{p_k}(B)$ for $k \geq k_0$. It follows that

$$\left| \int_B |u_k|^{p_k} \, dx - \int_B |u|^{p_k} \, dx \right| \leq \int_B p_k |u + \theta u_k|^{p_k - 1} |u_k - u| \, dx$$

$$\leq (p + \varepsilon) \left(\int_B |u + \theta u_k|^{p_k} \, dx \right)^{\frac{p_k - 1}{p_k}} \left(\int_B |u_k - u|^{p_k} \, dx \right)^{\frac{1}{p_k}}$$

$$\leq (p + \varepsilon) \left(\|u\|_{p_k} + \|u_k\|_{p_k} \right)^{p_k - 1} \left(\int_B |u_k - u|^{p_k} \, dx \right)^{\frac{1}{p_k}} \to 0$$

as $k \to +\infty$. It is clear that

$$\int_B |u|^{p_k} \, dx - \int_B |u|^p \, dx \to 0 \text{ as } k \to +\infty.$$

Thus,

$$\int_B |u_k|^{p_k} \, dx \to \int_B |u|^p \, dx.$$

Similarly, we also can obtain that

$$\int_B m^+(x)|u_k|^{p_k}\,dx \to \int_B m^+(x)|u|^p\,dx$$

and

$$\int_B m^-(x)|u_k|^{p_k}\,dx \to \int_B m^-(x)|u|^p\,dx,$$

where $m^+(x) = \max\{m(x), 0\}$, $m^-(x) = -\min\{m(x), 0\}$. Therefore,

$$
\begin{aligned}
\int_B m(x)|u_k|^{p_k}\,dx &= \int_B m^+(x)|u_k|^{p_k}\,dx - \int_B m^-(x)|u_k|^{p_k}\,dx \\
&\to \int_B m^+(x)|u|^p\,dx - \int_B m^-(x)|u|^p\,dx \\
&= \int_B m(x)|u|^p\,dx.
\end{aligned}
\tag{2.9}
$$

We note that (2.6) and (2.7) imply that

$$\mu_1(p_k)\int_B m(x)|u_k|^{p_k}\,dx = 1 \tag{2.10}$$

for all $k \in \mathbb{N}$. Thus letting k go to $+\infty$ in (2.10) and using (2.9), we find

$$\liminf_{j \to +\infty} \mu_1(p_k)\int_B m(x)|u|^p\,dx = 1. \tag{2.11}$$

On the other hand, since $u_k \rightharpoonup u$ in $W_{r,0}^{1,p-\varepsilon}(B)$, from (2.8) we obtain that

$$\|\nabla u\|_{p-\varepsilon}^{p-\varepsilon} \le \liminf_{k \to +\infty} \|\nabla u_k\|_{p-\varepsilon}^{p-\varepsilon} \le |B|^{\frac{\varepsilon}{p}}.$$

Now, letting $\varepsilon \to 0^+$ and applying Fatou's Lemma we find

$$\|\nabla u\|_p^p \le 1. \tag{2.12}$$

Hence $u \in W_r^{1,p}(B)$, here $W_r^{1,p}(B)$ denotes the radially symmetric subspace of $W^{1,p}(B)$. We claim that actually $u \in W_{r,0}^{1,p}(B)$. Indeed, we know that $u \in W_{r,0}^{1,p-\varepsilon}(B)$ for each $0 < \varepsilon < \varepsilon_0$. For $\phi \in C_{r,c}^\infty(\mathbb{R}^N)$ it is easy to see that

$$\left| \int_B u \frac{\partial \phi}{\partial x_i}\,dx \right| \le \|\nabla u\|_{p-\varepsilon}\|\phi\|_{(p-\varepsilon)'}, \ i = 1, \ldots, N.$$

Then, letting $\varepsilon \to 0^+$ we obtain that

$$\left| \int_B u \frac{\partial \phi}{\partial x_i}\,dx \right| \le \|\nabla u\|_p\|\phi\|_{(p)'}, \ i = 1, \ldots, N,$$

where $p' = p/(p-1)$. Since ϕ is arbitrary, from Proposition IX-18 of [3] we find that $u \in W_{r,0}^{1,p}(B)$, as desired.

Finally, combining (2.11) and (2.12) we obtain that

$$\liminf_{j \to +\infty} \mu_1(p_k)\int_B m(x)|u|^p\,dx \ge \int_B |\nabla u|^p\,dx.$$

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This and the variational characterization of $\mu_1(p)$ imply (2.5) and hence (2.3). This concludes the proof of the lemma. ∎

Using Remark 2.1, Theorem 2.1 and Theorem 2.2, we shall show that all eigenvalue functions $\mu_k^{\pm} : (1, +\infty) \to \mathbb{R}$, $2 \le k \in \mathbb{N}$ are continuous.

Theorem 2.3. *For each $2 \le k \in \mathbb{N}$, the eigenvalue function $\mu_k^{\nu} : (1, +\infty) \to \mathbb{R}$ is continuous.*

Proof. Let u_k^{ν} be an eigenfunction corresponding to $\mu_k^{\nu}(p)$. By Theorem 2.1 and Remark 2.1, we know that u has exactly $k-1$ simple zeros in I, i.e., there exist $c_{k,1}, \ldots, c_{k,k-1} \in I$ such that $u(c_{k,1}) = \cdots = u(c_{k,k-1}) = 0$. For convenience, we set $c_{k,0} = 0$, $c_{k,k} = 1$, $J_i = (c_{k,i-1}, c_{k,i})$ and $B_i = \{x \in B \,|\, c_{k,i-1} < |x| < c_{k,i}\}$ for $i = 1, \ldots, k$. Let $\mu_1^{\nu}(p, m/J_i, J_i)$ denote the first positive or negative eigenvalue of the restriction of problem (2.1) on J_i for $i = 1, \ldots, k$.

We note that Lemma 3 of [2] also holds for (2.1). It follows that $\mu_k^{\nu}(p) = \mu_1^{\nu}(p, m/J_i, J_i)$ for $i = 1, \ldots, k$. Using similar proof as Theorem 2.2, we can show that $\mu_1^{\nu}(p, m/J_i, J_i)$ is continuous with respect to p for $i = 1, \ldots, k$. Therefore, $\mu_k^{\nu}(p)$ is also continuous with respect to p. ∎

Finally, we give a key lemma that will be used in Section 4. Firstly, as an immediate consequence of Lemma 4.1 of [9], we obtain the following Sturm type comparison theorem.

Lemma 2.1. *Let $b_2(r) > b_1(r) > 0$ for $r \in (0,1)$ and $b_i(r) \in L^{\infty}(0,1)$, $i = 1, 2$. Also let u_1, u_2 are solutions of*

$$\left(r^{N-1}\varphi_p(u')\right)' + b_i(r)r^{N-1}\varphi_p(u) = 0, \; i = 1, 2,$$

respectively. If u_1 has k zeros in $(0,1)$, then u_2 has at least $k+1$ zeros in $(0,1)$.

Let
$$I^+ := \left\{r \in \overline{I} \,|\, m(r) > 0\right\}, \quad I^- := \left\{r \in \overline{I} \,|\, m(r) < 0\right\}.$$

Lemma 2.2. *Assume $m \in M(I)$. Let $\widehat{I} = [a, b]$ be such that $\widehat{I} \subset I^+$ and*

$$meas \, \widehat{I} > 0.$$

Let $g_n : \overline{I} \to (0, +\infty)$ be continuous function and such that

$$\lim_{n \to +\infty} g_n(r) = +\infty \quad uniformly \; on \; \widehat{I}.$$

Let $y_n \in E$ be a solution of the equation

$$\left(r^{N-1}\varphi_p(y_n')\right)' + r^{N-1}m(r)g_n(r)\varphi_p(y_n) = 0, \; r \in (0,1).$$

Then the number of zeros of $y_n|_{\widehat{I}}$ goes to infinity as $n \to +\infty$.

Proof. After taking a subsequence if necessary, we may assume that

$$m(r)g_{n_j}(r) \ge \lambda_j, \quad r \in \widehat{I},$$

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as $j \to +\infty$, where λ_j is the j-th eigenvalue of the following problem

$$
\begin{cases}
\left(r^{N-1}\varphi_p(u'(r))\right)' + \lambda r^{N-1}\varphi_p(u(r)) = 0, \ r \in I, \\
u'(0) = u(1) = 0.
\end{cases}
$$

Let φ_j be the corresponding eigenvalue of λ_j. It is easy to check that the number of zeros of $\varphi_j|_{\hat{I}}$ goes to infinity as $j \to +\infty$. By Lemma 2.1, one has that the number of zeros of $y_n|_{\hat{I}}$ goes to infinity as $n \to +\infty$. It follows the desired results. ∎

3 Unilateral global bifurcation phenomena for (1.2)

If $m(r) \equiv 1$, Del Pino and Elgueta [10] established the global bifurcation theory for one dimensional p-Laplacian eigenvalue problem. Peral [27] got the global bifurcation theory for p-Laplacian eigenvalue problem on the unite ball. In [9], Del Pino and Manásevich obtained the global bifurcation from the principle eigenvalue for p-Laplacian eigenvalue problem on the general domain. If $m(r) \geq 0$ and is singular at $r = 0$ or $r = 1$, Lee and Sim [20] also established the bifurcation theory for one dimensional p-Laplacian eigenvalue problem. However, if $m(r)$ changes sign, there are a few paper involving in the bifurcation theory for p-Laplacian eigenvalue problem. In this Section, we shall study the unilateral global bifurcation phenomena for N-dimensional p-Laplacian eigenvalue problem with sign-changing weight in the radial case.

Let $Y = L^1(0,1)$ with its usual normal $||\cdot||_{L^1}$ and $E = \left\{u \in C^1(\overline{I})\big|u'(0) = u(1) = 0\right\}$ with the norm

$$
\|u\| = \max_{r \in \overline{I}} |u(r)| + \max_{r \in \overline{I}} |u'(r)|.
$$

Considering the following auxiliary problem

$$
\begin{cases}
-\left(r^{N-1}|u'|^{p-2}u'\right)' = r^{N-1}h(r), & \text{a.e. } r \in I, \\
u'(0) = u(1) = 0
\end{cases}
\tag{3.1}
$$

for a given $h \in Y$. By a solution of problem (3.1), we understand a function $u \in E$ with $r^{N-1}\varphi_p(u')$ absolutely continuous which satisfies (3.1).

We have known that for every given $h \in Y$ there is a unique solution u to the problem (3.1) (see [9]). Let $G_p(h)$ denote the unique solution to (3.1) for a given $h \in Y$. It is well known that $G_p : Y \to E$ is continuous and compact (see [9, 27]).

Define $T_\mu^p(u) = G_p\left(\mu m(r)\varphi_p(u(r))\right)$. Let $\Psi_{p,\mu}$ be defined in E by

$$
\Psi_{p,\mu}(u) = u - T_\mu^p(u),
$$

where μ is a positive parameter. It is no difficult to show that $\Psi_{p,\mu}$ is a nonlinear compact perturbation of the identity. Thus the Leray-Schauder degree $\deg\left(\Psi_{p,\mu}, B_r(0), 0\right)$ is well-defined for arbitrary r-ball $B_r(0)$ and $\mu \neq \mu_k^\nu$.

Firstly, we can compute $\deg\left(\Psi_{2,\mu}, B_r(0), 0\right)$ for any $r > 0$ as follows.

Lemma 3.1. *For $r > 0$, we have*

$$
\deg\left(\Psi_{2,\mu}, B_r(0), 0\right) = \begin{cases}
1, & \text{if } \mu \in \left(\mu_1^-(2), \mu_1^+(2)\right), \\
(-1)^k, & \text{if } \mu \in \left(\mu_k^+(2), \mu_{k+1}^+(2)\right), \ k \in \mathbb{N}, \\
(-1)^k, & \text{if } \mu \in \left(\mu_{k+1}^-(2), \mu_k^-(2)\right), \ k \in \mathbb{N}.
\end{cases}
$$

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Proof. We divide the proof into two cases.

Case 1. $\mu \geq 0$.

Since G_2 is compact and linear, by [8, Theorem 8.10] and Theorem 2.1 with $p = 2$,

$$\deg\left(\Psi_{2,\mu}, B_r(0), 0\right) = (-1)^{m(\mu)},$$

where $m(\mu)$ is the sum of algebraic multiplicity of the eigenvalues μ of (2.1) satisfying $\mu^{-1}\mu_k^+ < 1$. If $\mu \in [0, \mu_1^+(2))$, then there are no such μ at all, then

$$\deg\left(\Psi_{2,\mu}, B_r(0), 0\right) = (-1)^{m(\mu)} = (-1)^0 = 1.$$

If $\mu \in \left(\mu_k^+(2), \mu_{k+1}^+(2)\right)$ for some $k \in \mathbb{N}$, then

$$\left(\mu_j^+(2)\right)^{-1}\mu > 1, \quad j \in \{1, \cdots, k\}.$$

This together with Theorem 2.1 implies

$$\deg\left(\Psi_{2,\mu}, B_r(0), 0\right) = (-1)^k.$$

Case 2. $\mu < 0$.

In this case, we consider a new sign-changing eigenvalue problem

$$\begin{cases} \left(r^{N-1}u'\right)' + \hat{\mu}\hat{m}(r)r^{N-1}u = 0, \ r \in I, \\ u'(0) = u(1) = 0, \end{cases}$$

where $\hat{\mu} = -\mu$, $\hat{m}(r) = -m(r)$. It is easy to check that

$$\hat{\mu}_k^+(2) = -\mu_k^-(2), \quad k \in \mathbb{N}.$$

Thus, we may use the result obtained in *Case 1* to deduce the desired result. ∎

As far as the general p is concerned, we can compute the extension of the Leray-Schauder degree defined in [4] by the deformation along p.

Lemma 3.2. (i) *Let* $\left\{\mu_k^+(p)\right\}_{k\in\mathbb{N}}$ *be the sequence of positive eigenvalues of (2.1). Let* μ *be a constant with* $\mu \neq \mu_k^+(p)$ *for all* $k \in \mathbb{N}$. *Then for arbitrary* $r > 0$,

$$\deg\left(\Psi_{p,\mu}, B_r(0), 0\right) = (-1)^\beta,$$

where β *is the number of eigenvalues* $\mu_k^+(p)$ *of problem (2.1) less than* μ.

(ii) *Let* $\{\mu_k^-(p)\}_{k\in\mathbb{N}}$ *be the sequence of negative eigenvalues of (2.1). Consider* $\mu \neq \mu_k^-(p)$, $k \in \mathbb{N}$, *then*
$$\deg\left(\Psi_{p,\mu}, B_r(0), 0\right) = (-1)^\beta, \ \forall r > 0,$$

where β *is the number of eigenvalues* $\mu_k^-(p)$ *of problem (2.1) larger than* μ.

Proof. We shall only prove the case $\mu > \mu_1^+(p)$ since the proof for the other cases are similar. We also only give the proof for the case $p > 2$. Proof for the case $1 < p < 2$ is similar. Assume that $\mu_k^+(p) < \mu < \mu_{k+1}^+(p)$ for some $k \in \mathbb{N}$. Since the eigenvalues depend

continuously on p, there exists a continuous function $\chi : [2, p] \to \mathbb{R}$ and $q \in [2, p]$ such that $\mu_k^+(q) < \chi(q) < \mu_{k+1}^+(q)$ and $\lambda = \chi(p)$. Define

$$\Upsilon(q, u) = u - G_q\left(\chi(q)m(r)\varphi_q(u)\right).$$

It is easy to show that $\Upsilon(q, u)$ is a compact perturbation of the identity such that for all $u \neq 0$, by definition of $\chi(q)$, $\Upsilon(q, u) \neq 0$, for all $q \in [2, p]$. Hence the invariance of the degree under homo-topology and Lemma 3.1 imply

$$\deg\left(\Psi_{p,\mu}, B_r(0), 0\right) = \deg\left(\Psi_{2,\mu}, B_r(0), 0\right) = (-1)^k.$$

∎

Define the Nemitskii operator $H : \mathbb{R} \times E \to Y$ by

$$H(\mu, u)(r) := \mu m(r)\varphi_p(u(r)) + g(r, u(r); \mu).$$

Then it is clear that H is continuous (compact) operator and problem (1.2) can be equivalently written as

$$u = G_p \circ H(\mu, u) := F(\mu, u).$$

F is completely continuous in $\mathbb{R} \times E \to E$ and $F(\mu, 0) = 0$, $\forall \mu \in \mathbb{R}$.

Using the similar method to prove [5, Theorem 2.1] with obvious changes, we may obtain the following result.

Theorem 3.1. *Assume (1.3) holds and $m \in M(I)$, then from each $(\mu_k^\nu, 0)$ it bifurcates an unbounded continuum $\mathcal{C}_k^\nu$ of solutions to problem (1.2), with exactly $k-1$ simple zeros, where μ_k^ν is the eigenvalue of problem (2.1).*

In what follows, we use the terminology of Rabinowitz [29]. Let S_k^+ denote the set of functions in E which have exactly $k-1$ interior nodal (i.e. non-degenerate zeros) in I and are positive near $t = 0$, and set $S_k^- = -S_k^+$, and $S_k = S_k^+ \cup S_k^-$. They are disjoint and open in E. The following global bifurcation result is a generalization of Theorem 3.2 of [5]. The essential idea is similar to the proof of Theorem 3.2 of [5]

Theorem 3.2. *Assume (1.3) holds and $m \in M(I)$, then there are two distinct unbounded continua, $(\mathcal{C}_k^\nu)^+$ and $(\mathcal{C}_k^\nu)^-$, consisting of the bifurcation branch $\mathcal{C}_k^\nu$. Moreover, for $\sigma \in \{+, -\}$, we have*

$$(\mathcal{C}_k^\nu)^\sigma \subset \left(\{(\mu_k^\nu, 0)\} \cup (\mathbb{R} \times S_k^\sigma)\right).$$

4   Existence of nodal solutions of (1.5)

In this Section, we shall investigate the existence and multiplicity of nodal solutions to the problem (1.5) under the linear growth condition on f.

Firstly, we suppose that

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(H_1) $f \in C(\mathbb{R}, \mathbb{R})$ *with* $f(s)s > 0$ *for* $s \neq 0$;

(H_2) *there exists* $f_0 \in (0, +\infty)$ *such that*

$$f_0 = \lim_{|s| \to 0} \frac{f(s)}{\varphi_p(s)};$$

(H_3) *there exists* $f_\infty \in (0, +\infty)$ *such that*

$$f_\infty = \lim_{|s| \to +\infty} \frac{f(s)}{\varphi_p(s)}.$$

Let $\mu_k^{\pm}$ be the k-th positive or negative eigenvalue of (2.1). Applying Theorem 3.2, we shall establish the existence of nodal solutions of (1.5) follows.

Theorem 4.1. *Assume* (H_1), (H_2) *and* (H_3) *hold and* $m \in M(I)$. *Assume that for some* $k \in \mathbb{N}$, *either*

$$\gamma \in \left(\frac{\mu_k^+(p)}{f_\infty}, \frac{\mu_k^+(p)}{f_0} \right) \cup \left(\frac{\mu_k^-(p)}{f_0}, \frac{\mu_k^-(p)}{f_\infty} \right)$$

or

$$\gamma \in \left(\frac{\mu_k^+(p)}{f_0}, \frac{\mu_k^+(p)}{f_\infty} \right) \cup \left(\frac{\mu_k^-(p)}{f_\infty}, \frac{\mu_k^-(p)}{f_0} \right).$$

Then (1.5) has two solutions u_k^+ *and* u_k^- *such that* u_k^+ *has exactly* $k-1$ *zeros in* I *and is positive near 0, and* u_k^- *has exactly* $k-1$ *zeros in* I *and is negative near 0.*

Proof. We only prove the case of $\gamma > 0$. The case of $\gamma < 0$ is similar. Consider the problem

$$\begin{cases} \left(r^{N-1} \varphi_p(u') \right)' + \mu \gamma r^{N-1} m(r) f(u) = 0, \ r \in I, \\ u'(0) = u(1) = 0. \end{cases} \tag{4.1}$$

Let $\zeta \in C(\mathbb{R})$ be such that $f(u) = f_0 \varphi_p(u) + \zeta(u)$ with $\lim_{|u| \to 0} \zeta(u)/\varphi_p(u) = 0$. Hence, the condition (1.3) holds. Using Theorem 3.2, we have that there are two distinct unbounded continua, $(\mathcal{C}_k^\nu)^+$ and $(\mathcal{C}_k^\nu)^-$, consisting of the bifurcation branch $\mathcal{C}_k^\nu$ from $(\mu_k^\nu(p)/\gamma, 0)$, such that

$$(\mathcal{C}_k^\nu)^\sigma \subset (\{(\mu_k^\nu, 0)\} \cup (\mathbb{R} \times S_k^\sigma)).$$

It is clear that any solution of (4.1) of the form $(1, u)$ yields a solutions u of (1.5). We shall show that $(C_k^+)^\sigma$ crosses the hyperplane $\{1\} \times E$ in $\mathbb{R} \times E$. To this end, it will be enough to show that $(C_k^+)^\sigma$ joins $(\mu_k^+(p)/\gamma f_0, 0)$ to $(\mu_k^+(p)/\gamma f_\infty, +\infty)$. Let $(\eta_n, y_n) \in (C_k^+)^\sigma$ satisfy $\eta_n + \|y_n\| \to +\infty$. We note that $\eta_n > 0$ for all $n \in \mathbb{N}$ since $(0,0)$ is the only solution of (4.1) for $\mu = 0$ and $(C_k^+)^\sigma \cap (\{0\} \times E) = \emptyset$.

Case 1: $\mu_k^+(p)/f_\infty < \gamma < \mu_k^+(p)/f_0$. In this case, we only need to show that

$$\left(\frac{\mu_k^+(p)}{\gamma f_\infty}, \frac{\mu_k^+(p)}{\gamma f_0} \right) \subseteq \{\mu \in \mathbb{R} | (\mu, u) \in (C_k^+)^\sigma\}.$$

We divide the proof into two steps.

Step 1: We show that if there exists a constant $M > 0$ such that $\eta_n \subset (0, M]$ for $n \in \mathbb{N}$ large enough, then $(C_k^+)^\sigma$ joins $(\mu_k^+(p)/\gamma f_0, 0)$ to $(\mu_k^+(p)/\gamma f_\infty, +\infty)$.

In this case it follows that $\|y_n\| \to +\infty$. Let $\xi \in C(\mathbb{R})$ be such that $f(u) = f_\infty \varphi_p(u) + \xi(u)$. Then $\lim_{|u| \to +\infty} \frac{\xi(u)}{\varphi_p(u)} = 0$. Let $\widetilde{\xi}(u) = \max_{0 \le |s| \le u} |\xi(s)|$. Then $\widetilde{\xi}$ is nondecreasing and

$$\lim_{u \to +\infty} \frac{\widetilde{\xi}(u)}{|u|^{p-1}} = 0. \tag{4.2}$$

We divide the equation

$$\left(r^{N-1}\varphi_p(y_n')\right)' - \mu_n \gamma r^{N-1} m(r) f_\infty \varphi_p(y_n) = \mu_n \gamma r^{N-1} m(r) \xi(y_n)$$

by $\|y_n\|$ and set $\overline{y}_n = y_n/\|y_n\|$. Since $\overline{y}_n$ is bounded in E, after taking a subsequence if necessary, we have that $\overline{y}_n \rightharpoonup \overline{y}$ for some $\overline{y} \in E$. Moreover, from (4.2) and the fact that $\widetilde{\xi}$ is nondecreasing, we have that

$$\lim_{n \to +\infty} \frac{\xi(y_n(r))}{\|y_n\|^{p-1}} = 0$$

since

$$\frac{\xi(y_n(r))}{\|y_n\|^{p-1}} \le \frac{\widetilde{\xi}(|y_n(r)|)}{\|y_n\|^{p-1}} \le \frac{\widetilde{\xi}(\|y_n(r)\|_\infty)}{\|y_n\|^{p-1}} \le \frac{\widetilde{\xi}(\|y_n(r)\|)}{\|y_n\|^{p-1}}.$$

By the continuity and compactness of G_p, it follows that

$$-\left(r^{N-1}\varphi_p(\overline{y}')\right)' = \overline{\mu} \gamma r^{N-1} m(r) f_\infty \varphi_p(\overline{y}),$$

where $\overline{\mu} = \lim_{n \to +\infty} \mu_n$, again choosing a subsequence and relabeling if necessary.

We claim that $\overline{y} \in \left(C_k^+\right)^\sigma$.

It is clear that $\|\overline{y}\| = 1$ and $\overline{y} \in \overline{\left(\mathcal{C}_k^+\right)^\sigma} \subseteq \left(\mathcal{C}_k^+\right)^\sigma$ since $\left(\mathcal{C}_k^+\right)^\sigma$ is closed in $\mathbb{R} \times E$. Therefore, by Theorem 2.1, $\overline{\mu} \gamma f_\infty = \mu_k^+(p)$, so that $\overline{\mu} = \mu_k/\gamma f_\infty$. Therefore $(C_k^+)^\sigma$ joins $\left(\mu_k^+(p)/\gamma f_0, 0\right)$ to $\left(\mu_k^+(p)/\gamma f_\infty, +\infty\right)$.

Step 2: We show that there exists a constant M such that $\mu_n \in (0, M]$ for $n \in \mathbb{N}$ large enough.

On the contrary, we suppose that $\lim_{n \to +\infty} \mu_n = +\infty$. Since $(\eta_n, y_n) \in (C_k^+)^\sigma$, it follows that

$$\left(r^{N-1}\varphi(y_n')\right)' + \gamma \eta_n r^{N-1} m(r) f(y_n) = 0.$$

Let

$$0 < \tau(1, n) < \cdots < \tau(k, n) = 1$$

be the zeros of y_n in $\overline{I}$. Then, after taking a subsequence if necessary,

$$\lim_{n \to +\infty} \tau(l, n) := \tau(l, \infty), \qquad l \in \{1, \cdots, k-1\}.$$

It follows that either there exists at least one $l_0 \in \{1, \cdots, k-1\}$ such that

$$\tau(l_0, \infty) < \tau(l_0 + 1, \infty) \text{ or } \tau(1, \infty) = 1.$$

Notice that Lemma 2.2 and the fact y_n has exactly $k-1$ simple zeros in $\overline{I}$ yield

$$\left\{\left[\cup_{l=1}^{k-1}\left(\tau(l, \infty), \tau(l+1, \infty)\right)\right] \cup (0, \tau(1, \infty))\right\} \cap I^+ = \emptyset,$$

which implies that

$$\left\{ \left[\cup_{l=1}^{k-1} \big(\tau(l, \infty), \tau(l+1, \infty) \big) \right] \cup (0, \tau(1, \infty)) \right\} \subseteq (I \setminus I^+).$$

Therefore,

$$\text{meas}\,(I \setminus I^+) \geq \text{meas}\,\left\{ \left[\cup_{l=1}^{k-1} \big(\tau(l, \infty), \tau(l+1, \infty) \big) \right] \cup (0, \tau(1, \infty)) \right\} = 1.$$

However, this contradicts (H_2): $0 < \text{meas}\,(I \setminus I^+) < 1$.

Case 2: $\mu_k^+(p)/f_0 < \gamma < \mu_k^+(p)/f_\infty$. In this case, we have that

$$\frac{\mu_k^+(p)}{\gamma f_0} < 1 < \frac{\mu_k^+(p)}{\gamma f_\infty}.$$

Assume that $(\eta_n, y_n) \in \big(C_k^+ \big)^\sigma$ is such that $\lim_{n \to +\infty} (\eta_n + \|y_n\|) = +\infty$. In view of *Step 2* of *Case 1*, we have known that there exists $M > 0$, such that for $n \in \mathbb{N}$ sufficiently large, $\eta_n \in (0, M]$. Applying the same method used in *Step 1* of *Case 1*, after taking a subsequence and relabeling if necessary, it follows that

$$(\eta_n, y_n) \to \left(\frac{\mu_k^+(p)}{\gamma f_\infty}, +\infty \right) \quad \text{as } n \to +\infty.$$

Thus, $(C_k^+)^\sigma$ joins $\big(\mu_k^+(p)/\gamma f_0, 0 \big)$ to $\big(\mu_k^+(p)/\gamma f_\infty, +\infty \big)$. ∎

Using the similar proof with the proof Theorem 4.1, we can obtain the more general results as follows.

Theorem 4.2. *Assume (H_1), (H_2) and (H_3) hold and $m \in M(I)$. Assume that for some $k, n \in \mathbb{N}$ with $k \leq n$, either*

$$\gamma \in \left(\frac{\mu_n^+(p)}{f_\infty}, \frac{\mu_k^+(p)}{f_0} \right) \cup \left(\frac{\mu_k^-(p)}{f_0}, \frac{\mu_n^-(p)}{f_\infty} \right)$$

or

$$\gamma \in \left(\frac{\mu_n^+(p)}{f_0}, \frac{\mu_k^+(p)}{f_\infty} \right) \cup \left(\frac{\mu_k^-(p)}{f_\infty}, \frac{\mu_n^-(p)}{f_0} \right).$$

Then (1.5) has $n - k + 1$ pairs solutions u_j^+ and u_j^- for $j \in \{k, \cdots, n\}$ such that u_j^+ has exactly $j - 1$ zero in I and is positive near 0, and u_j^- has exactly $j - 1$ zero in I and is negative near 0.

Remark 4.1. We would like to point out that Theorem 1.1 of [24] is the corollary of Theorem 5.1 even in the case of $p = 2$ and $N = 1$.

Remark 4.2. We also note that Theorem 4.1 and Theorem 4.2 is valid for the problems on annular domain because it can be convert the equivalent one-dimensional problems.

An open problem. When $m \geq 0$, using Lemma 2.1, we can easily get that (1.5) has no nontrivial solution if $\gamma m f(u)/u$ not cross any eigenvalue of (1.6). Therefore, we conjecture that (1.5) has no nontrivial solution if

$$\mu_k^+(p) < \frac{f(s)}{\varphi_p(s)} < \mu_{k+1}^+(p) \text{ or } \mu_k^-(p) > -\frac{f(s)}{\varphi_p(s)} > \mu_{k+1}^-(p) \text{ for } s \neq 0.$$

5 One-sign solutions for (1.7)

In this Section, based on the bifurcation result of Drábek and Huang [11], we shall study the existence of one-sign solutions for problem (1.7). From now on, for simplicity, we write $X := W_0^{1,p}(\Omega)$.

The main results of this section are the following:

Theorem 5.1. *Let (H_1), (H_2) and (H_3) hold, and $m \in M(\Omega)$. Assume that either*

$$\gamma \in \left(\frac{\mu_1^+(p)}{f_\infty}, \frac{\mu_1^+(p)}{f_0} \right) \cup \left(\frac{\mu_1^-(p)}{f_0}, \frac{\mu_1^-(p)}{f_\infty} \right)$$

or

$$\gamma \in \left(\frac{\mu_1^+(p)}{f_0}, \frac{\mu_1^+(p)}{f_\infty} \right) \cup \left(\frac{\mu_1^-(p)}{f_\infty}, \frac{\mu_1^-(p)}{f_0} \right).$$

then problem (1.7) possesses at least a positive and a negative solution.

Remark 5.1. By the $C^{1,\alpha}$ $(0 < \alpha < 1)$ regularity results for quasilinear elliptic equations with p-growth condition [21], $u \in C^{1,\alpha}(\overline{\Omega})$ for any solution u of (1.7) since f is continuous and subcritical.

In order to prove Theorem 5.1, we consider the following eigenvalue problem

$$\begin{cases} -\mathrm{div}\,(\varphi_p(\nabla u)) = \mu\gamma m(x)f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where μ is a parameter. Let $\zeta \in C(\mathbb{R})$ be such that

$$f(u) = f_0\varphi_p(u) + \zeta(u)$$

with $\lim_{|u|\to 0} \zeta(u)/\varphi_p(u) = 0$. Let us consider

$$\begin{cases} -\mathrm{div}\,(\varphi_p(\nabla u)) = \mu\gamma m(x)f_0\varphi_p(u) + \mu\gamma m(x)\zeta(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

as a bifurcation problem from the trivial solution $u \equiv 0$.

Let

$$\mathbb{S}^+ = \left\{ u \in C^{1,\alpha}(\overline{\Omega}) \big| u(x) > 0, \text{ for all } x \in \Omega \right\} \text{ and } \mathbb{S}^- = \left\{ u \in C^{1,\alpha}(\overline{\Omega}) \big| u(x) < 0, \text{ for all } x \in \Omega \right\}.$$

Applying Theorem 4.4 and 4.5 of [11] to (5.2), we can obtain the following unilateral global bifurcation result, which plays a fundamental role in our study.

Lemma 5.1. *Let $\nu \in \{+, -\}$. There are two distinct unbounded continua, $\mathcal{C}_+^\nu$ and $\mathcal{C}_-^\nu$, consisting of the bifurcation branch $\mathcal{C}^\nu$ from $(\mu_k^\nu(p), 0)$. Moreover, for $\sigma \in \{+, -\}$, we have*

$$\mathcal{C}_\sigma^\nu \subset \left(\{(\mu_1(p), 0)\} \cup (\mathbb{R} \times \mathbb{S}^\sigma) \right).$$

We use Lemma 6.1 to prove the main results of this section.

Proof of Theorem 5.1. Since the proof is similar to that of Theorem 4.1, we only give a rough sketch of the proof. We only prove the case of $\gamma > 0$. The case of $\gamma < 0$ is similar. It is clear that any solution of (5.1) of the form $(1, u)$ yields a solution u of (1.7). We shall show $\mathcal{C}_\sigma^+$ crosses the hyperplane $\{1\} \times X$ in $\mathbb{R} \times X$. To this end, it will be enough to show that $\mathcal{C}_\sigma^+$ joins $\left(\mu_1^+(p)/\gamma f_0, 0\right)$ to $\left(\mu_1^+(p)/\gamma f_\infty, +\infty\right)$.

Let $(\mu_n, y_n) \in \mathcal{C}_\sigma^+$ where $y_n \not\equiv 0$ satisfies $\mu_n + \|y_n\|_X \to +\infty$. We note that $\mu_n > 0$ for all $n \in \mathbb{N}$ since $(0,0)$ is the only solution of (5.1) for $\mu = 0$ and $\mathcal{C}_\sigma^+ \cap (\{0\} \times X) = \emptyset$.

Case 1: $\mu_1^+(p)/f_\infty < \gamma < \mu_1^+(p)/f_0$.

In this case, we only need to show that

$$\left(\frac{\mu_1^+(p)}{\gamma f_\infty}, \frac{\mu_1^+(p)}{\gamma f_0}\right) \subseteq \left\{\mu \in \mathbb{R} \,\middle|\, (\mu, u) \in \mathcal{C}_\sigma^+\right\}.$$

We divide the proof into two steps.

Step 1: We show that if there exists a constant $M > 0$ such that $\eta_n \subset (0, M]$ for $n \in \mathbb{N}$ large enough.

In this case it follows that $\|y_n\| \to +\infty$. Similar to the proof of Theorem 4.1, we divide the equation

$$-\operatorname{div}(\varphi_p(\nabla y_n)) - \mu_n \gamma m(x) \varphi_p(y_n) = \mu_n \gamma m(x) \xi(y_n)$$

by $\|y_n\|_{C^{1,\alpha}(\overline{\Omega})}$ and set $\overline{y}_n = y_n/\|y_n\|_{C^{1,\alpha}(\overline{\Omega})}$. Since $\overline{y}_n$ is bounded in $C^{1,\alpha}(\overline{\Omega})$, after taking a subsequence if necessary, we have that $\overline{y}_n \rightharpoonup \overline{y}$ for some $\overline{y} \in C^{1,\alpha}(\overline{\Omega})$ and $\overline{y}_n \to \overline{y}$ in $C(\overline{\Omega})$. Using the similar method to the proof of Theorem 4.1, we can obtain

$$\lim_{n \to +\infty} \frac{\xi(y_n(t))}{\|y_n\|_{C^{1,\alpha}(\overline{\Omega})}^{p-1}} = 0 \text{ as } n \to +\infty.$$

By the compactness of $R_p : L^\infty(\Omega) \to X$ (see [9]), we obtain

$$-\operatorname{div}(\varphi_p(\nabla \overline{y})) - (\overline{\mu} m(x) \varphi_p(\overline{y})) = 0,$$

where $\overline{\mu} = \lim\limits_{n \to +\infty} \mu_n$, again choosing a subsequence and relabeling if necessary. The rest proof of this step is the same as the proof of Theorem 4.1.

Step 2: We show that there exists a constant M such that $\mu_n \in (0, M]$ for $n \in \mathbb{N}$ large enough.

On the contrary, we suppose that $\lim_{n \to +\infty} \mu_n = +\infty$. Since $(\mu_n, y_n) \in \mathcal{C}_\sigma^+$, it follows that

$$\operatorname{div}(\varphi_p(\nabla y_n)) + \gamma \mu_n m(x) \frac{f(y_n)}{\varphi(y_n)} \varphi(y_n) = 0 \text{ in } \Omega^+,$$

where $\Omega^+ = \{x \in \Omega \,|\, m(x) > 0\}$. By Theorem 2.6 of [1], we have y_n must change sign in Ω^+, which contradicts Lemma 5.1. The rest proof of is similar to the proof of Theorem 4.1.

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